

A Problem in Probability

Suppose that we perform an experiment that involves randomly throwing darts at a dartboard. We then record where they land. We would like to determine the probability of a dart landing in a given region of the dartboard.

To record the position of each dart, we will use polar coordinates. Recall that every point in the

Cartesian plane has polar coordinates (r, θ) , where θ is the angle that the ray emanating from the origin $(0, 0)$ through the point makes with the positive x -axis and r is the distance to the point from the origin (see **fig. 1**). By convention, we assume that $0 \leq \theta < 2\pi$. If we consider the dartboard to be a circle of radius 1 centered at the origin, then all points of the dartboard except for the origin are specified by $0 < r < 1$. To sidestep any ambiguity, we also use the convention that the origin is represented by $(0, 0)$ in polar coordinates.

To compute *empirical probabilities* from this experiment, we might break up the dartboard into regions, count the number of times a dart lands in a given region, and then divide that number by the total number of trials. We assume that we can throw the darts in a truly random fashion, so that there is an equal probability of the dart landing anywhere on the board, but that the dart must land on the board. The empirical results should produce a close approximation of the *theoretical probability* of the dart landing in any given region. But then a question arises: How do we compute the theoretical probability that a dart will land in a given region of the dartboard?

Finding theoretical probabilities in a discrete setting is fairly simple: Determine the complete

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Edited by **J. Kevin Colligan**, jkcolligan@verizon.net
RABA Center of SRA International, Columbia, MD 21045

Dan Kalman, kalman@american.edu
American University, Washington, DC 20016

Virginia Stallings, vstalli@american.edu
American University, Washington, DC 20016

Jeffrey Wanko, wankojj@muohio.edu
Miami University, Oxford, OH 45056

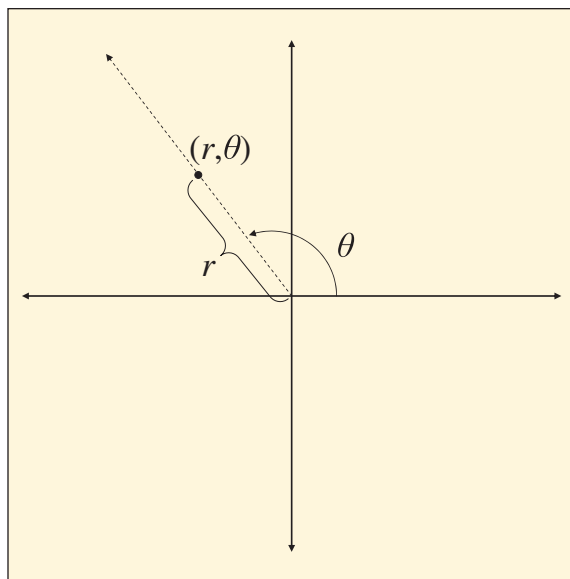


Fig. 1 Polar coordinates can also be used to represent a point in the plane.

sample space of the probability experiment and then divide the number of occurrences of a desired outcome by the total number of outcomes. For example, when computing the theoretical probability of rolling a die and obtaining 1 or 3, we simply notice that 2 of the 6 possible outcomes in the sample space $\{1, 2, 3, 4, 5, 6\}$ will give us the desired result; hence, the probability is $2/6 = 1/3$. This approach may be referred to as an “equally likely outcomes” method, and it results in a discrete uniform distribution.

This method cannot be used for the dartboard problem, however, because in our geometric conception of the problem there are infinitely many points where the dart can land. Instead, we use our intuition and define probability geometrically—as, for example, in Bannon’s analysis of three ways to break a stick (Bannon 2009). The basic idea is that the probability of a randomly chosen point falling into a specified region is proportional to the size of the region. For the two-dimensional geometry of the dartboard problem, size is represented by area. Analogously, size corresponds to length when we choose points on a line and to volume when we choose points in space.

In the case of the dartboard, which we shall call D , a uniform probability distribution would assign equal probabilities to regions within D of equal area, thus implying that the probability of a dart landing in a certain region can be found by dividing the area of the region by the area of the entire circle. This method extends to other geometric figures, but we use the unit disk here. For instance, the theoretical probability of the dart landing in C , the central disk of radius $1/2$ inside the circle of radius 1 , would be as follows:

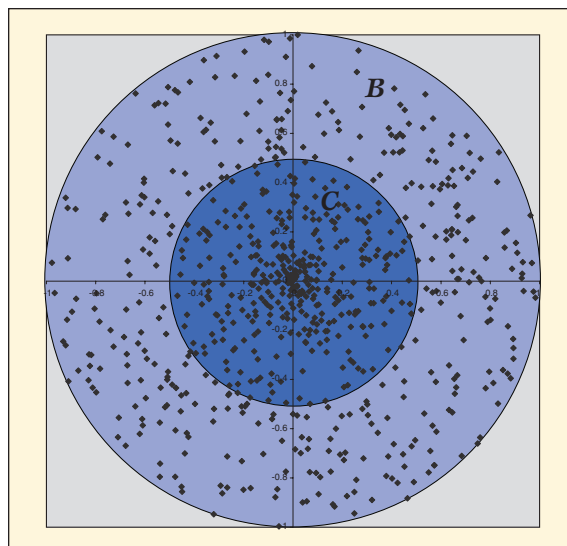


Fig. 2 Uniformly choosing r and θ and plotting these polar coordinates produces results that do not seem uniform.

$$P_{Area}(C) = \frac{Area(C)}{Area(D)} = \frac{\pi^2/4}{\pi^2} = \frac{1}{4}$$

As we will see, this is not the only way to calculate the probability of a dart landing in C . A different approach may lead to a different answer. To emphasize that the value $1/4$ is based on area, we denote it as $P_{Area}(C)$.

We can simulate this experiment by using computer software. For example, Microsoft Excel has a simple random number generator that we can use to choose points uniformly within an interval—that is, the probabilities of choosing points from subintervals of the same length will be equal. At first glance, we might think that we can apply this method to the dartboard problem in the following way.

Use a uniform distribution to randomly choose an r between 0 and 1 , then choose θ between 0 and 2π , and finally plot the resulting point using these numbers as polar coordinates. A simple experiment using Microsoft Excel reveals the plot of random points chosen in this way (see **fig. 2**). However, the distribution of points does not appear to be uniform here; the points seem to be concentrated near the origin.

Let’s look again at the inner disk C of radius $1/2$. When we choose points in C , the θ -coordinate can be anything from 0 to 2π . As for r , if we randomly choose a number between 0 and 1 , the probability that it will lie between 0 and $1/2$ is $1/2$. Because choosing such an r would result in our plotted point being in C , the probability of a dart landing in C is exactly $1/2$. We denote this by $P_{Polar}(C) = 1/2$. In fact, the likelihood of a dart landing in C is equal to the likelihood of a dart landing in the outer band B , because the probability of a dart landing in B is the same as the probability of picking a value of r

between $1/2$ and 1 , which is also $1/2$. But does this not contradict our usual intuition about probabilities? Using the basic area formulas above, we determined that the probability of choosing a point in C should be $1/4$. Something seems awry here.

In the brief mathematical exploration that follows, we will attempt to get to the bottom of this seeming paradox. In the process, we will unearth some interesting assumptions and connections inherent in the study of probability.

A CASUAL ANALYSIS OF THE PROBLEM

Let's return to the manner by which we chose a point in C . We randomly chose r and θ using one-dimensional uniform distributions. We can envision this as a way to choose a point (r, θ) from the open rectangle $R = [0, 1) \times [0, 2\pi)$, treating r and θ as if they were Cartesian coordinates (see **fig. 3a**). Moreover, it is easy to see that this approach results in a uniform distribution of points in the rectangle.

This selection process means that the probabilities of choosing points from same-area subregions within the rectangle are the same. Notice that the probability of choosing a point in the left half of this rectangle is $1/2$ and that choosing a point here is equivalent to the point landing in C . If we use L to denote this half-rectangle and P_{Area} to denote the uniform probability in the rectangle R , then we are saying that $P_{Area}(L) = 1/2 = P_{Polar}(C)$.

What have we achieved? This particular way of choosing points in the dartboard D renders the dartboard somehow "equivalent" to the rectangle.

Here we use the term "equivalent" loosely, in the sense that one can find a mapping M of the rectangle onto the dartboard that transfers the uniform probability of subregions of R to their images in D . This mapping is the polar coordinate transformation, which is described by the following set of parametric equations:

$$\begin{aligned} x &= r \cos \theta \text{ and } y = r \sin \theta \text{ for} \\ (r, \theta) &\in R \text{ and } (x, y) \in D \end{aligned} \quad (1)$$

Equivalently,

$$M(r, \theta) = (r \cos \theta, r \sin \theta). \quad (2)$$

By picking various values for r and θ and seeing where they land in D , we see that the pair of equations in (1) essentially takes R and squeezes and stretches it unevenly (along with applying some counterclockwise rotation) so as to form the unit disk D (see **fig. 4**).

And here we can see why what seemed to be a paradox arose earlier. The transformation from the rectangle to the circle distorts areas. Two regions with equal areas in the rectangle can correspond to

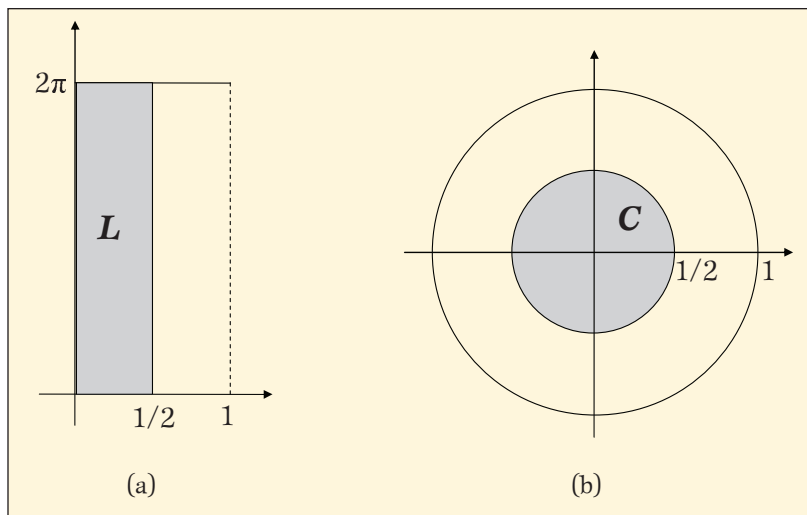


Fig. 3 A point with Cartesian coordinates (r, θ) is in the shaded rectangle L (a) if and only if the same (r, θ) are polar coordinates of a point in shaded disk C (b).

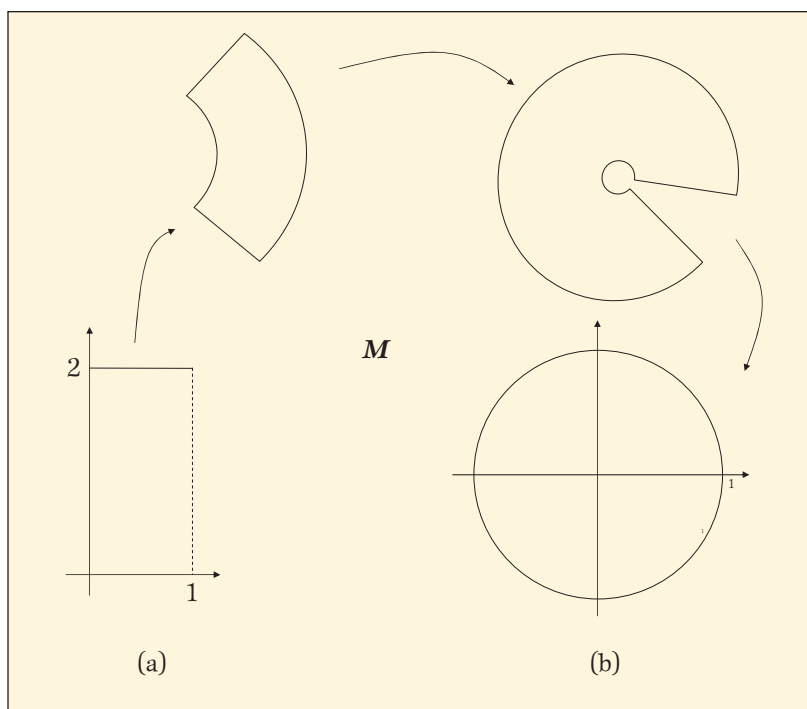


Fig. 4 If the rectangle (a) were made of malleable rubber, it could be de-formed into the circle (b) as shown.

regions of unequal areas in the circle. Therefore, although we see a uniform distribution of points in the rectangle, we should not expect to observe a uniform distribution of points in the circle.

Take, for example, the region C , where $P_{Polar}(C)$ equals $1/2$ rather than the $1/4$ we would expect by computing areas in the circle. Points near the left edge of the rectangle become squeezed down into a region near the origin. In fact, all the points along the left edge are compressed in D into a single point, the origin. One could think of D as being heavily "weighted" there, probabilistically speaking. This situation stands in stark contrast to a uniform

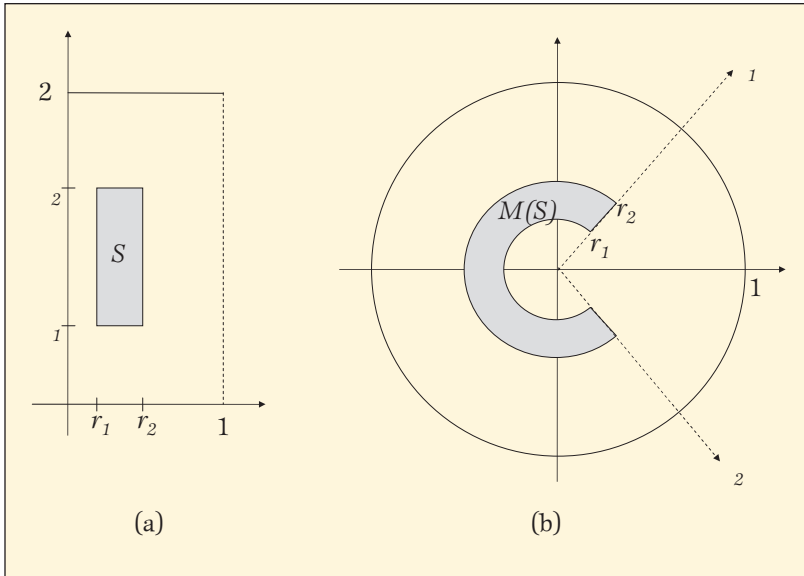


Fig. 5 Transformation M carries the rectangle S (a) onto the C-shaped region $M(S)$ (b).

distribution on D , which would place an equal “weight” on regions of the same area in D . This “distortion” and nonuniformity of “weight” brought about by our mapping cause an equal likelihood of landing in C as in the outer band B , when we choose the polar coordinates of the points uniformly.

We can quantify these phenomena by taking a look at how M affects the areas of subrectangles in the rectangle R . The results can be extended to regions of other shapes.

Let S be a subrectangle defined by all points (r, θ) where $0 < r_1 \leq r \leq r_2 < 1$ and $0 < \theta_1 \leq \theta \leq \theta_2 < 2\pi$. Then, as shown in **figure 5**, the image $M(S)$ has area

$$\text{Area}(M(S)) = \frac{1}{2}(r_2^2 - r_1^2)(\theta_2 - \theta_1).$$

This mapping shows how subrectangles of equal area in R may have images under M that have different areas. For instance, the left and right halves of R clearly have the same area, but their images are the central region C and the outer band B , which have unequal areas. Or, for a more visual example, imagine sliding S (in **fig. 5**) to the right. Its image then moves radially outward in the circle D , increasing in area in the process.

SOLVING THE PROBLEM OF POINT PICKING

The fact that r is squared when finding the areas of such regions results in the distortion and shrinking closer to the origin described above. One might wonder if we could make up for such distortion and shrinking when finding areas by choosing points in a different way. As a matter of fact, suppose that we picked our points as follows.

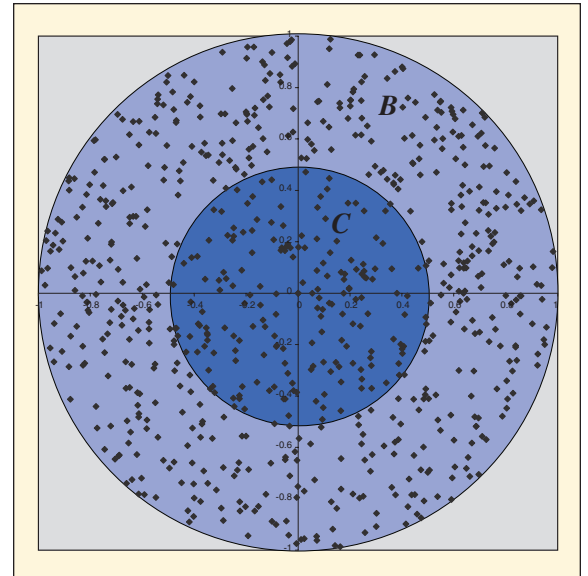


Fig. 6 “Choosing” points with the mapping T produces a more uniform distribution.

As before, we continue to choose θ uniformly in the interval $[0, 2\pi)$, but now we choose r uniformly from the interval $[0, 1]$, let $s = \sqrt{r}$, and then plot the point with polar coordinates (s, θ) . This way of choosing points can be thought of as a new transformation $T(r, \theta)$ where

$$T(r, \theta) = (\sqrt{r} \cos \theta, \sqrt{r} \sin \theta).$$

Recall that the subrectangle S in R is defined by all points (r, θ) where $0 < r_1 \leq r \leq r_2 < 1$ and $0 < \theta_1 \leq \theta \leq \theta_2 < 2\pi$. Now, under the mapping T , we see that

$$\begin{aligned} \text{Area}(T(S)) &= \frac{1}{2}(s_2^2 - s_1^2)(\theta_2 - \theta_1) \\ &= \frac{1}{2}(r_2 - r_1)(\theta_2 - \theta_1) \\ &= \frac{1}{2}\text{Area}(S). \end{aligned}$$

In particular, under T , rectangular regions in R with the same area are mapped to regions with the same area in D .

Another simple Microsoft Excel experiment that uses the new mapping T results in the distribution of randomly chosen points, as shown in **figure 6**. It appears that this distribution is uniform and that we have solved the problem of simulating the random dart-throwing experiment.

This entire discussion has focused solely on rectangular regions and their images in the dartboard under the mappings M and T . To consider more general regions, we need to make use of the tools of calculus. That discussion can be found online (go to www.nctm.org/mt).

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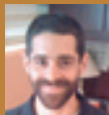


For a consideration of more general regions, which requires the use of calculus, go to the NCTM Web site: www.nctm.org/mt.



YUICHI HANDA, yhanda@csuchico.edu, and CHRISTOPHER YAKES, cyakes@csuchico.edu, are colleagues at California State University in Chico. Handa's interest is phenomenology; Yakes's is real and complex analysis. Both are interested in the professional development of K-12

mathematics teachers. YUICHI HANDA; CHRISTOPHER YAKES



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A Problem in Probability

OTHER REGIONS

The print version of this issue's "Delving Deeper" has focused solely on rectangular regions and their images in the dartboard under the mappings M and T . To consider more general regions, we turn to the tools of calculus.

Calculus allows us to determine areas of very general regions in the plane and thus to find probabilities of regions using integrals. For example, in the case of the rectangle R described in the print article, the probability of choosing a point (r, θ) uniformly from any region E in R is $\text{Area}(E)/\text{Area}(R)$. In turn,

$$\text{Area}(E) = \iint_E dr d\theta.$$

As a particular case, let E be the rectangle R_t defined by $0 \leq r \leq t$ and $0 \leq \theta \leq 2\pi$. We see that

$$P_{\text{Area}}(R_t) = \frac{1}{2\pi} \iint_{R_t} dr d\theta = \frac{1}{2\pi} \int_0^{2\pi} \int_0^t dr d\theta = \frac{2\pi \cdot t}{2\pi} = t. \quad (3)$$

(Equations [1] and [2] appear in the print version of this month's "Delving Deeper.")

Turning to the case of the dartboard D , we can express the (uniform) probability of a region E as $\text{Area}(E)/\text{Area}(D) = \text{Area}(E)/\pi$. Again, the area of E can be expressed as an integral, but this time we use x and y as the variables because we envision D as situated in a Cartesian xy -plane. For example, if E is a smaller disk D_t of radius t between 0 and 1, the probability of choosing a point that lies in D_t is

$$P_{\text{Area}}(D_t) = \frac{\text{Area}(D_t)}{\pi} = \frac{1}{\pi} \iint_{D_t} dx dy. \quad (4)$$

This integral is challenging to compute because the region D_t is not nearly as simple as a rectangle. (What would the limits of integration be?) However, a theorem from calculus allows us to convert

this integral into a simpler integral involving polar coordinates. More generally, it specifies how to introduce a transformation of variables in a double integral. Our specific case involves the transformation from rectangular to polar coordinates, but we state the theorem in the general case, as follows: Given a suitable mapping M from a region A in the uv -plane to a region B in the xy -plane, there is a function $J_M(u, v)$ such that

$$\iint_A J_M(u, v) du dv = \iint_B dx dy. \quad (5)$$

The technically correct statement of the theorem has to include certain assumptions about M , such as differentiability. Here, we are content to note that the two specific mappings we are concerned with satisfy the technical requirements. The function J_M is found by taking the absolute value of the determinant of the matrix that consists of the partial derivatives of the mapping M . This matrix is referred to as the *Jacobian matrix* of M .

In our situation, the plane being represented in polar coordinates (r, θ) can be thought of as the uv -plane mentioned in the theorem and $M(r, \theta) = (r \cos \theta, r \sin \theta)$. We can compute that $J_M = r$. Thus, we see that using the polar-coordinate transformation M to transfer points uniformly chosen from the rectangle R to the dartboard D introduces an extra factor of r into the formula for computing area and, hence, into the formula for probabilities. Indeed, combining equations (4) and (5), we have

$$P_{\text{Area}}(D_t) = \frac{1}{\pi} \iint_{D_t} dx dy = \frac{1}{\pi} \iint_{R_t} J_M(r, \theta) dr d\theta = \frac{1}{\pi} \int_0^{2\pi} \int_0^t r dr d\theta = t^2. \quad (6)$$

Here, we have used the fact that the region R_t is mapped by M onto the region D_t . This formula gives the correct (geometric) probability of a dart

landing in the region D_t . This probability is different from the probability that in our original discussion was “transferred” over from the rectangle by mapping M (shown in (3) to be t).

The last integral in (6) looks similar to the integral in (3), except for the factor of $2r$ that appears in (6). And herein lies the rub: Given that the limits of integration are the same in each case, the factor of $2r$ in the case of the disk suppresses the contribution of points closer to the origin, because this is where r is relatively small. However, in the case of the rectangle, the factor of r is missing, reflecting the uniform distribution of all points regardless of how small r might be. This result corresponds to what we discovered earlier.

Consider now our second method of picking points, which involves choosing a point (r, θ) in R and applying the “altered” polar-coordinate transformation $T(r, \theta)$ where

$$T(r, \theta) = (\sqrt{r} \cos \theta, \sqrt{r} \sin \theta).$$

Here, the Jacobian function $J_T(r, \theta)$ is constant, simply equal to $1/2$. Again, consider an arbitrary disk D , centered at the origin of radius t in D . The set

$$R_{t^2} = [0, t^2] \cdot [0, 2\pi)$$

gets mapped to D_t by T . We compute the probability of a dart landing in the set D_t as

$$\begin{aligned} P_{Area}(D_t) &= \frac{1}{\pi} \iint_{D_t} dx dy \\ &= \frac{1}{\pi} \iint_{R_{t^2}} J_T(r, \theta) dr d\theta \\ &= \frac{1}{\pi} \int_0^{2\pi} \int_0^{t^2} \frac{1}{2} dr d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{t^2} dr d\theta = t^2 = P_{Area}(R_{t^2}). \end{aligned}$$

This computation shows that in some sense T is the “correct” transformation to use to transfer uniformly chosen points to the dartboard while preserving the intrinsic uniform distribution on both the rectangle R and the dartboard D . These methods can be extended to more general subregions of R and their images in the dartboard.

POINT PICKING

What is the upshot of our little excursion, you may ask? It is—plain and simple—that *probability* is dependent on how the points are picked. The interested reader should investigate Bertrand’s paradox, in which the simple question of finding the probability that a random chord of a circle will be longer than a side of an inscribed equilateral triangle has three different solutions, all depending on how the chords of the circle are chosen. Indeed, problems such as dartboard point picking and Bertrand’s paradox serve to challenge our assumptions about probability. Investigating them can further our understanding of these and other mathematical ideas.

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